

# Dirac membrane and hadrons

Maciej Trzetrzelewski \*

NORDITA,  
Roslagstullsbacken 23, 106 91 Stockholm,  
Sweden

## Abstract

In  $c = 1$  units the product (mass  $\times$  radius) for the neutron and the proton is about  $4.7\hbar$  assuming their radii equal to 1fm. We show that the corresponding products for the Dirac neutral and charged membrane coincide and are equal  $1.6\hbar$ .

## 1 Introduction

In 1962 Dirac, trying to explain the value of the muon mass, considered an idea that the electron could be modeled by a conducting, elastic membrane of spherical topology [1] (at the same time he also considered a neutral membrane in the presence of the gravitational field [2] - the finite size of the electron was originally considered much earlier by Lorentz, Abraham, Bucherer and Langevin, see [3] and references therein). Assuming the Lagrangian  $L = L_{EM} + L_{mem.}$ , where  $L_{EM}$  is a usual term for the electromagnetic field and  $L_{mem.}$  is given by the membrane world-volume, it can be shown that for the spherically symmetric case with the radial coordinate  $\rho$ , the Hamiltonian of the model (in  $c = 1$  units) is given by

$$H_r = \sqrt{-\hbar^2 \partial_\rho^2 + \omega^2 \rho^4} + \frac{e^2}{2\rho} \quad (1)$$

where  $\omega/4\pi$  is the membrane tension and  $e$  is the electric charge. Using the Bohr-Sommerfeld quantization method, one finds that the first

---

\*e-mail: maciej.trzetrzelewski@gmail.com

excitation of the membrane corresponds to the energy  $\approx 53m_e$ , where  $m_e$  is the mass of the electron. Dirac notes that in order to get a closer value to the experimental  $m_\mu \approx 207m_e$  one would presumably have to introduce spin into the theory by performing the square root appearing in  $H_r$ . While this might be true, in this paper we focus on another possible application of Dirac's membrane i.e. the effective description of hadrons. We find a surprisingly good agreement between the product (mass $\times$ radius) for charged/neutral membrane and the experimental values for the proton/neutron respectively.

## 2 Dirac's model

Among all closed (compact, without the boundary) objects coupled to the electromagnetic field  $A_\mu$  in  $D = 4$  spacetime dimensions, membranes play a special role ensuring that  $A_\mu$  is nowhere singular. The reason for this lying in the fact that such object split  $\mathbb{R}^3$  into two disconnected regions: the interior and the exterior. Equations of motion for membranes are nonlinear hence quite complicated and unlike in the case of points or strings very little is known about the exact solutions (for some examples see [4]). As for the quantum theory, the existing method of quantizing the bosonic membrane [5] is highly non-trivial resulting in a certain quantum mechanical model with matrix degrees of freedom. When the size of a matrix is taken to infinity the precise description of the quantum membrane is obtained. In spite of these complications recently there has been a considerable progress in understanding the bosonic membrane theory [6].

Since closed membranes in  $\mathbb{R}^3$  provide a natural split of space into the interior and the exterior there exists a proffered curvilinear system  $x^\mu$  in spacetime and a function  $f(x)$  such that the equation  $f(x) = 0$  describes a membrane and equations  $f(x) > 0$ ,  $f(x) < 0$  describe a region outside or inside the membrane, respectively. It is convenient to fix the curvilinear system such that  $x^1 = f(x)$  and choose  $\sigma^0 = x^0 =: \tau$ ,  $\sigma^1 = x^2$ ,  $\sigma^2 = x^3$  for the three variables  $\sigma^\alpha$ ,  $\alpha = 0, 1, 2$  - the internal parametrization of the membrane world-volume. The action for a membrane coupled to the electromagnetic field considered by Dirac [1] is (in  $c = 1$  units)

$$S = S_{EM} + S_{mem.}, \quad (2)$$

$$S_{EM} = -\frac{1}{16\pi} \int_{x^1 > 0} J g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F^{\rho\sigma} d^4x, \quad F_{\mu\nu} = \partial_{[\mu} A_{\nu]},$$

$$S_{mem.} = -\frac{\omega}{4\pi} \int_{x^1=0} M dx^0 dx^2 dx^3$$

where  $g^{\mu\nu}$  is the metric corresponding to the curvilinear system  $x^\mu$  (concretely Dirac takes the induced metric  $g_{\mu\nu} = \partial_\mu y^\Lambda \partial_\nu y_\Lambda$ ,  $\Lambda = 0, 1, 2, 3$ , where  $y^\Lambda$  are rectilinear and orthogonal),  $J = \sqrt{-\det g_{\mu\nu}}$  and  $M = J\sqrt{-g^{11}}$ . The coupling is due to the factor  $J$  appearing in both  $S_{EM}$  and  $S_{mem.}$ . Varying (2) with respect to  $y^\Lambda$  one arrives at the equations of motion which for the spherically symmetric case  $x^1 = r - \rho$ ,  $x^2 = \theta$ ,  $x^3 = \phi$  (so that the surface is given by  $x^1 = 0$ ) give

$$\frac{d}{dt} \frac{\dot{\rho}}{\sqrt{1-\dot{\rho}^2}} = \frac{e^2}{2\omega\rho^4} - \frac{2}{\rho\sqrt{1-\dot{\rho}^2}}. \quad (3)$$

The balance between the repulsive electromagnetic forces and the attractive ones, due to the positive membrane tension, is when  $\dot{\rho} = 0$  hence  $a^3 = e^2/4\omega$  where  $a$  is the radius of the electron. On the other hand the total energy of a system at rest  $E = e^2/2\rho + \beta\rho^2$  (minimal in the equilibrium provided  $\beta = \omega$ ) is equal to both  $3e^2/4a$  and  $m_e$  when  $\rho = a$ . Therefore one concludes that  $a = 3e^2/4m_e = 0.75r_e$  where  $r_e$  is the classical electron radius,  $r_e \approx 2.8\text{fm}$  (the value  $a = 2.1\text{fm}$  is of course not realistic as it is bigger then the charge radius of the proton  $0.87\text{fm}$  [7] - according to Dehmelt the radius of the electron could be of order  $10^{-7}\text{fm}$  [8]). Considering small oscillation about the equilibrium one finds the corresponding frequency to be  $\sqrt{6}/a$  hence the energy of one quantum would be  $h\nu = \sqrt{6}\hbar/a = 448m_e$ .

To improve the analysis one proceeds to the hamiltonian formalism. While the details of this are quite involved the final answer for the spherically symmetric case turns out to be particularly simple given by (1). The complications are due to the choice of the coordinates  $x^\mu$  hence the loss of the explicit diffeomorphism invariance of the action. There exists a generally covariant formulation [9] in which one considers, in addition to  $S_{EM}$  and  $S_{mem.}$  written for an arbitrary  $x^\mu$  and  $\sigma^\alpha$ , an extra term proportional to  $e^\alpha \partial_\alpha X^\mu A_\mu$  where  $e^\alpha$  is the current-charge density on the membrane. One then shows that all Dirac's findings, in particular the hamiltonian (1), can be obtained in a less elaborate way keeping the explicit diffeomorphism invariance.

Using the Bohr-Sommerfeld quantization condition one obtains an approximation which can be written as

$$\frac{m_{Dirac}}{m_e} = \frac{1}{3} \left( \frac{32\sqrt{\pi}\Gamma(7/4)}{\alpha\Gamma(1/4)} n \right)^{\frac{2}{3}}$$

where  $\alpha = e^2/\hbar c$  is the fine structure constant, obtaining

$$\frac{m_{Dirac}}{m_e} \approx 52.4, \quad 83.1, \quad 109.0, \quad 132.0, \quad \dots$$

for consecutive values of  $n$ .

This approximation can be improved as observed in [10] by noting that in the  $\alpha \rightarrow 0$  limit the coulomb term in (1) acts like an infinite wall and hence one should apply different boundary conditions in the Bohr-Sommerfeld quantization procedure. As a result one obtains

$$\frac{m_{improved}}{m_e} \approx 43.3, \quad 76.1, \quad 102.9, \quad 126.1, \quad \dots$$

which agree with the numerical values also found in [10].

## 2.1 Further investigation

Let us rewrite the hamiltonian  $H_r$  in (1) in dimensionless variable  $x = \rho/a$

$$H_r = \frac{4m_e}{3\alpha} \left( \sqrt{-\partial_x^2 + \frac{\alpha^2}{16}x^4} + \frac{\alpha}{2x} \right) \quad (4)$$

where we used  $\omega = \frac{e^2}{4a^3}$ ,  $a = \frac{3e^2}{4m_e}$ .

While finding an analytic expression for the square root of the positive differential operator  $H$  is in general difficult the calculation of the matrix elements of the square root  $\sqrt{H}$  is possible using the standard procedure by diagonalizing  $H$  and taking  $\sqrt{H} = U^T \sqrt{E} U$  where  $U$  is s.t.  $H = U^T E U$ ,  $E = \text{diag}(E_n)$ . Applying numerical techniques (for details see the Appendix - we use a different technique compared to [10]) we find that this prescription gives

$$\frac{m_{our}}{m_e} \approx 43.6, \quad 76.3, \quad 103.0, \quad 126.5, \quad \dots$$

which are consistent with the results of [10].

In order to find the radius of the first two excitations (let us denote the corresponding wavefunctions by  $\psi_\mu$  for  $E_1 = 43.6m_e$  and  $\psi_\tau$  for  $E_2 = 76.3m_e$ ) we use their probability densities (Figure 1). It follows that  $|\psi_\mu(\rho)|^2$  has a maximum for  $\rho \approx 6.7a$  while  $|\psi_\tau(\rho)|^2$  has two maxima for  $\rho \approx 3.8a$  and  $\rho \approx 11.9a$  so that the probability density  $|\psi_\tau(\rho)|^2$  will be viewed in  $\mathbb{R}^3$  as two concentric membranes.

It is somewhat interesting to consider the generalization of Dirac action to the case of two (or more) concentric membranes. We will

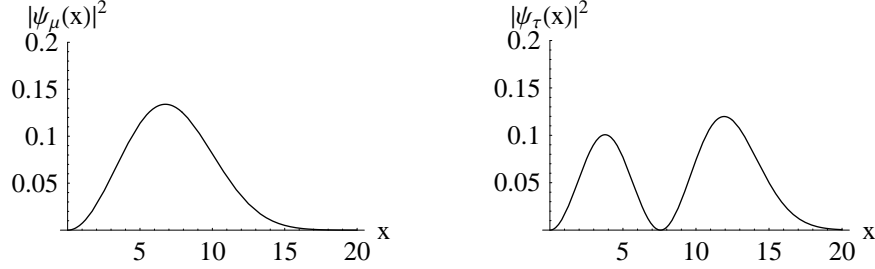


Figure 1: Probability densities  $|\psi_\mu(x)|^2$  and  $|\psi_\tau(x)|^2$ .

assume that the total charge  $e$  is on the exterior membrane. If the tension of the inner, neutral shell is  $k\omega$ ,  $k > 0$  the corresponding hamiltonian of this system would be

$$H_{r\rho} = \sqrt{-\partial_r^2 + k^2\omega^2 r^4} + \sqrt{-\partial_\rho^2 + \omega^2 \rho^4} + \frac{e^2}{2\rho}. \quad (5)$$

The eigenvalues of this hamiltonian are lifted compared to  $H_r$ . One can choose  $k$  so that its first excitation is equal to the mass of the muon (this takes place for  $k \approx 63$ ) however the second excitation is not even close to the mass of the taon. Of course one could consider three concentric membranes and fit the additional tension so that the mass of the taon appears but such model would be just a fit.

### 3 Reinterpretation

Perhaps the result  $m_\mu = 43.6m_e$  could be improved by introducing spin in some clever way into the theory. However it seems that this membrane model works surprisingly well if we identify charged membrane with the proton and the neutral membrane with the neutron.

Let us start with the case of the neutral membrane. The discreteness of the hamiltonian (1) is not due to the Coulomb term. Classically, without the electromagnetic field the membrane will collapse (i.e.  $\rho = 0$  after a finite time -  $\rho(\tau)$  can be obtained from the equations of motion which imply  $\ddot{\rho}\rho = 2(\dot{\rho}^2 - 1)$  solved by the Jacobi sine function) but the operator

$$H_{neut.} = \sqrt{-\hbar^2 \partial_\rho^2 + \Omega^2 \rho^4} = (\hbar^2 \Omega)^{1/3} \sqrt{-\partial_y^2 + y^4} \quad (6)$$

where  $y = \rho/(\hbar/\Omega)^{1/3}$  is dimensionless, itself is discrete (since it is a square root of a discrete, positive definite operator) hence quantum

mechanically a free membrane will develop bound states. The natural unit of energy is now  $(\hbar^2\Omega)^{1/3}$  where the tension  $\Omega$  is not specified. The hamiltonian  $H_{neut.}$  has parity even and odd eigenstates. The parity even states have the maximum of the probability density for  $\rho = 0$  which corresponds to a membrane with 0 radius - a point. However the parity odd states have the maximum for  $\rho > 0$  for which the eigenvalues are

$$\frac{m_{neut.}}{(\hbar^2\Omega)^{1/3}} \approx 1.95, \quad 3.41, \quad 4.61, \quad 5.66 \dots$$

The probability density of the first excitation  $m_0 \approx 1.95(\hbar^2\Omega)^{1/3}$  has a maximum at  $\rho_0 \approx 0.82(\hbar/\Omega)^{1/3}$  (Figure 2) hence  $m_0\rho_0 \approx 1.6\hbar$ .

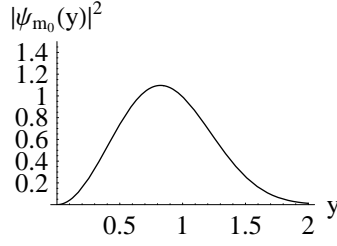


Figure 2: First excitation for a neutral membrane.

It is quite amusing that for the case of the neutron we have  $m_n\rho_n \approx 4.7\hbar$  (in  $c = 1$  units, we took  $\rho_n = 1\text{fm}$ ) in rough agreement with the result for the neutral membrane. Considering the fact that we do not mention quarks, gluons and the spin, this naive picture of a neutron as a neutral membrane works surprisingly well.

We can now do the analogous analysis for the charged membrane. The hamiltonian is the same as for the case of the electron only written in different units (cp. (4))

$$H_{charged} = (4\hbar^2\Omega/\alpha)^{1/3} \left( \sqrt{-\partial_z^2 + \frac{\alpha^2}{16}z^4} + \frac{\alpha}{2z} \right)$$

(where,  $z = \rho/(\alpha\hbar/4\Omega)^{1/3}$ ,  $\Omega$  can be related to the classical radius at the equilibrium). Due to the Coulomb term we must assume  $\psi(0) = 0$  for the wavefunctions hence all the excitations have nonzero radius. The eigenvalues are

$$\frac{m_{charged}}{(4\hbar^2\Omega/\alpha)^{1/3}} \approx 0.239, \quad 0.418, \quad 0.564, \quad 0.693, \dots$$

and the lowest energy state  $m_0 \approx 0.239(4\hbar^2\Omega/\alpha)^{1/3}$  is peaked at  $\rho_0 \approx 6.75(\alpha\hbar/4\Omega)^{1/3}$  (same as for  $\psi_\mu$ , see Figure 1) implying  $m_0\rho_0 \approx 1.6\hbar$ .

## 4 Conclusions

It is not a surprise that effective description of hadrons in terms of bag-like models will essentially give results consistent with experimental data. However what is striking about the Dirac membrane model is its huge simplicity compared to the actual processes taking place - no quarks, gluons, spin - one wonders why this model works at all? Moreover, the value of the only dimensional parameter of the theory, the tension  $\Omega$ , was nowhere used as it cancels out in the calculation. It seems that, at least for the purposes of calculating the product (mass  $\times$  radius), the spin can be neglected while the strong forces replaced by a  $2 + 1$  dimensional field theory - signaling a sort of holographic [12] behavior.

**Acknowledgment** I thank M. Kuźniak and P. O. Mazur for discussions and the correspondence as well as the Swedish Research Council and KTH for support.

## Appendix

In order to obtain numerically the eigenvalues of

$$H_x := \frac{4}{3\alpha} \left( \sqrt{\partial_x^2 + \frac{\alpha^2}{16}x^4} + \frac{\alpha}{2x} \right)$$

we use the cutoff method [11] which consists of calculating the matrix elements of  $H_x$  in some orthonormal basis (on  $[0, \infty)$  in our case), truncate the infinite matrix and then numerically diagonalize it. The spectra and the eigenvectors of the truncated matrices converge very quickly to their exact counterparts.

An important step in this method is the choice of the convenient basis (a common choice is the basis in the Fock space - not suitable here). Due to the  $1/x$  part in  $H_x$  the regularity of the wavefunction  $\psi(x)$  at the origin implies that  $\psi(x) \sim x^l$  for  $l \geq 1$ . For this reason we choose the orthonormal basis  $e_n(x)$ ,  $n = 1, 2, 3, \dots$  as

$$e_n(x) = \frac{1}{\sqrt{n(n+1)}} x L_{n-1}^{(2)}(x) e^{-x}, \quad \int_0^\infty e_n(x) e_m(x) dx = \delta_{nm}$$

where  $L_n^{(2)}(x)$  are generalized Laguerre polynomials. In this basis the explicit representation of operators  $K = -\frac{d^2}{dx^2}$ ,  $W = x^4$  and  $V = 1/x$  can be obtained with

$$K_{nm} := (e_n, K e_m), \quad W_{nm} := (e_n, W e_m), \quad V_{nm} := (e_n, V e_m).$$

At this point we introduce a cutoff  $N_{max}$ , consider a finite matrix  $h_{nm}^{(N_{max})} = K_{nm} + W_{nm}$ ,  $n, m \leq N_{max}$  and then numerically find its eigenvalues  $E_k^{(N_{max})}$  and the matrix  $U^{(N_{max})}$  s.t.

$$h^{(N_{max})} = U^{(N_{max}) T} E^{(N_{max})} U^{(N_{max})}, \quad E^{(N_{max})} = \text{diag}(E_k^{(N_{max})}).$$

The square root of  $h^{(N_{max})}$  can now be obtained and the matrix representation for the overall operator  $H_x$  is

$$H_x^{(N_{max})} = U^{(N_{max}) T} \sqrt{E}^{(N_{max})} U^{(N_{max})} + V^{(N_{max})}.$$

The spectrum of  $H_x^{(N_{max})}$  quickly converges to the exact values (Figure 3). To obtain the wavefunctions corresponding to  $E_n^{(N_{max})}$  we use the

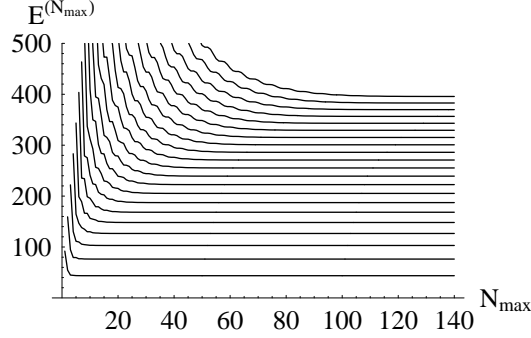


Figure 3: The convergence of the eigenvalues of  $H_x^{(N_{max})}$ .

eigenvectors  $v_n^{(N_{max})}$  of  $H_x^{(N_{max})}$  i.e.

$$\psi_n^{(N_{max})}(x) = \sum_{k=1}^{N_{max}} [v_n^{(N_{max})}]_k e_k(x).$$

The convergence of  $\psi_n^{(N_{max})}(x)$  to the exact  $\psi_n(x)$  is governed by the corresponding convergence of the eigenvectors. In practice the cutoff=10 already gives very accurate approximation for first excitations.



To find the eigenvalues of the parity even states of (6) we use the same numerical method but with help of a different, more convenient basis

$$f_n(x) = \frac{1}{\sqrt{2^{n-1}n!\sqrt{\pi}}} H_{2n-1}(x)e^{-x^2/2}, \quad \int_0^\infty f_n(x)f_m(x)dx = \delta_{nm}$$

where  $H_n(x)$  are Hermite polynomials.

## References

- [1] P. A. M. Dirac, *An Extensible Model of the Electron*, Proc. Roy. Soc. A268, (1962) 57-67.
- [2] P. A. M. Dirac, *Motion of an Extended Particle in the Gravitational Field*, in *Relativistic Theories of Gravitation*, Proceedings of a Conference held in Warsaw and Jablonna, July 1962, ed. L. Infeld, P. W. N. Publishers, 1964, Warsaw, 163-171; discussion 171-175.  
P. A. M. Dirac, *Particles of Finite Size in the Gravitational Field*, Proc. Roy. Soc. A270, (1962) 354-356.
- [3] A. K. Wróblewski, *Einstein and physics hundred years ago*, Acta Phys. Polon. B37 (2006) 1.  
A. K. Wróblewski, *Physics 1909: A portrait of the field hundred years ago*, Acta Phys. Polon. B41 (2010) 229.
- [4] R. W. Tucker, *Extended Particles and the Exterior Calculus*, Lectures given at the Rutherford Laboratory, Feb 1976.  
M. Bordemann, J. Hoppe, *The Dynamics of Relativistic Membranes I: Reduction to 2-dimensional Fluid Dynamics*, Phys. Lett. B317 (1993) 315.  
M. Bordemann, J. Hoppe, *The Dynamics of Relativistic Membranes II: Nonlinear Waves and Covariantly Reduced Membrane Equations*, Phys. Lett. B325 (1994) 359.  
J. Hoppe, *Some Classical Solutions of Relativistic Membrane Equations in 4 Space-Time Dimensions*, Phys. Lett. B329 (1994) 10-14.  
J. Hoppe, *Canonical 3+1 description of relativistic membranes*, arXiv:hep-th/9407103v2.

- J. Hoppe, *Conservation Laws and Formation of Singularities in Relativistic Theories of Extended Objects*, Phys. Lett. B335 (1994) 41, [arXiv:hep-th/9503069](#).
- J. Arnlin, J. Hoppe, S. Theisen, *Spinning membranes*, Phys. Lett. B599 (2004) 118-128.
- J. Hoppe,  *$U(1)$  invariant Membranes and Singularity Formation*, Compl. anal. oper. theory 3 (2009), 419-424, Birkhauser , Basel, ISSN 1661-8254, [arXiv:0805.4738v1](#).
- M. Trzetrzelewski, A. A. Zheltukhin, *Exact solutions for  $U(1)$  globally invariant membranes*, Phys. Lett. B679 (2009) 523-528, [arXiv:0903.5062](#).
- M. Trzetrzelewski, A. A. Zheltukhin,  *$U(1)$ -invariant membranes: the zero curvature formulation, Abel and pendulum differential equations*, J. Math. Phys. 51 (2010) 1.
- [5] J. Hoppe, *Quantum Theory of a Massless Relativistic Surface and a two dimensional bound state problem*, Ph.D. Thesis MIT, (1982).
- [6] M. Trzetrzelewski, *Spiky membranes*, Phys. Lett. B684 (2010) 256-261, [arXiv:0910.3870](#).
- J. Hoppe, *Fundamental Structures of  $M$ (brane) Theory*, [arXiv:1003.5189](#).
- J. de Woul, J. Hoppe, D. Lundholm, *Partial Hamiltonian reduction of relativistic extended objects in light-cone gauge*, JHEP 1101:031 (2011), [arXiv:1006.4714](#).
- J. Hoppe, *Matrix Models and Lorentz Invariance*, J. Phys. A44:055402 (2011), [arXiv:1007.5505](#).
- J. Hoppe,  *$M$ -brane dynamical symmetry and quantization*, [arXiv:1101.4334](#).
- J. Hoppe, M. Trzetrzelewski, *Lorentz-invariant membranes and finite matrix approximations*, [arXiv:1101.4403](#).
- [7] K. Nakamura et al. (Particle Data Group), J. Phys. G37 075021 (2010).
- [8] H. Dehmelt, *Experiments with an isolated subatomic particle at rest*, Rev. of Mod. Phys. 62, (1990) 525-530; Nobel Lecture.
- [9] A. O. Barut, M. Pavšič, *Dirac's shell model of the electron and the general theory of moving relativistic charged membranes*, Phys. Lett. B306 (1993) 49-54.

- A. O. Barut, *Strings and Membranes in Electron Theory*, Prepared for Gursey Memorial Conference I: On Strings and Symmetries, Istanbul, Turkey, 6-10 Jun 1994.
- [10] P. Gnädig, Z. Kunszt, P. Hasenfratz, J. Kutti, *Dirac's Extended Electron Model*, Ann. of Phys., 116 2 (1978) 380-407.
- [11] J. Wosiek, *Spectra of supersymmetric Yang-Mills quantum mechanics*, Nucl. Phys. B 644 (2002) 85, [arXiv:hep-th/0203116v1](#).  
M. Trzetrzelewski, *Quantum mechanics in a cut Fock space*, M.Sc. Thesis, Jagiellonian University, (2003).  
M. Trzetrzelewski, J. Wosiek, *Quantum systems in a cut Fock space*, Acta Phys. Polon. B 35 (2004) 1615, [arXiv:hep-th/0308007](#).  
M. Trzetrzelewski, *Quantum mechanics in a cut Fock space*, Acta Phys. Polon. B35 (2004) 2393, [arXiv:hep-th/0407059](#).
- [12] G. 't Hooft, *Dimensional Reduction in Quantum Gravity*, [arXiv:gr-qc/9310026](#).